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Calculus of Variations Tutorial:

(adapted from "Mathematics and Technology" by Rousseau and Saint-Aubin & John Strain's notes)

• **Introduction:**

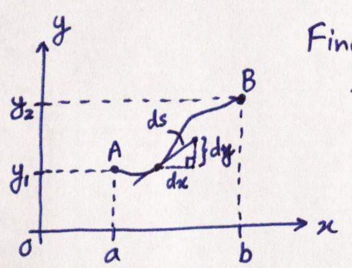
- branch of applied mathematics dealing with optimization over function spaces
- many applications to physics & engineering
- Used in Hamiltonian mechanics – bridge between Newtonian and quantum mechanics
- Recall Lagrange multiplier method:

$$f: \mathbb{R}^n \to \mathbb{R},\; g: \mathbb{R}^n \to \mathbb{R}$$

$$\left.\begin{array}{l} \min\limits_{x} f(x) \\ \text{s.t. } g(x) = c \end{array}\right\} \underbrace{\mathcal{L}(x,\lambda)}_{\text{Lagrangian}} = f(x) + \lambda(g(x)-c), \quad \underbrace{\nabla \mathcal{L} = 0}_{\text{stationary conditions}}$$

In variational calculus, we optimize over function spaces rather than $\mathbb{R}^n$.

- **Example 1:** (Shortest Path)



Find shortest path between A & B.

- Ans: Straight line → Δ inequality.

Formalism: Let $y = y(x) \Rightarrow$ Path parametrized by $(x, y(x))$, $x \in [a,b]$.

Let $\underbrace{I[y]}_{\substack{\text{functional} \\ \text{(function of function)}}}$ = length of path between A & B

$$I[y] \triangleq \int_A^B ds = \int_A^B \underbrace{\sqrt{dx^2+dy^2}}_{\text{Pythagoras' Thm}} = \int_a^b \sqrt{1+(y')^2}\, dx$$

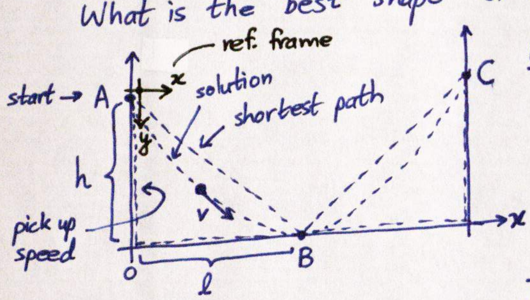
∴ Our problem is:

$$\min_{\substack{y(x):\\ y(a)=y_1,\, y(b)=y_2}} I[y] = \int_a^b \sqrt{1+(y')^2}\, dx \,.$$

} can try to solve this → DIRECT METHOD

- **Example 2:** (Brachistochrone – "shortest time") → posed by Johann Bernoulli as contest & solved by Newton, Leibniz, L'Hôpital, Johann & Jacob Bernoulli

What is the best "shape" of a skateboard ramp?



Want: Minimum time from A to B, powered only by gravity.

Let the path between A & B be $(x, y(x))$. Let $I[y]$ = total time.

Formalism: (Conservation of energy)

Let energy at A be $E = 0$ (stationary)

$$\underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic}} = \underbrace{mgy}_{\text{potential}} \quad \text{(for some pt between A \& B)} \Rightarrow v = \sqrt{2gy}$$

$$I[y] \triangleq \int_A^B dt = \int_A^B \frac{ds}{v} = \int_0^{\ell} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx = \frac{1}{\sqrt{2g}} \int_0^{\ell} \sqrt{\frac{1+(y')^2}{y}}\, dx$$

∴ Our problem is:

$$\min_{\substack{y(x):\\ y(0)=0,\, y(\ell)=h}} I[y] = \frac{1}{\sqrt{2g}} \int_0^{\ell} \sqrt{\frac{1+(y')^2}{y}}\, dx \,.$$

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- **Fundamental Problem of Calculus of Variations:**

Given a Lagrangian: $L : [a,b] \times \mathbb{R} \times \mathbb{R}$, $L(x, y, z)$

admissible functions: $C = \{ y : [a,b] \to \mathbb{R} \mid \underbrace{y(a) = y_1,\ y(b) = y_2}_{\text{boundary conditions}},\ y \text{ is } \underbrace{\text{twice differentiable}}_{\text{regularity conditions: Hölder derivatives}} \}$

cost function: $I[y] = \int_a^b L(x, y(x), y'(x))\, dx$ ← called action of physical system

the problem is: find extremal values

$$\min_{y \in C} I[y].$$

- **Euler-Lagrange Equations:** (Systematic / indirect method of solution)

- Thm: If $y_0 \in C$ minimizes $I[y]$ over C, then:

$$\boxed{\frac{\partial L}{\partial y}(x, y_0, y_0') - \frac{d}{dx}\left(\frac{\partial L}{\partial z}(x, y_0, y_0')\right) = 0}.$$

- only necessary conditions
- solution may be local optimum, or saddle pt, etc.

Fundamental Lemma of Calculus of Variations: (FLCV) $\int_a^b u(x) w(x)\, dx = 0$ for all $w \in C$ if and only if $u(x) = 0$. ← compare with finite case (vectors)

Pf: ($\Rightarrow$) Let $w = u$. $\int_a^b u(x)^2\, dx = 0 \Rightarrow u(x) = 0$. ▨

($\Leftarrow$) Obvious.

} can make this measure theoretic

Pf: Suppose y_0 minimizes $I[y]$ over C. Let $w : [a,b] \to \mathbb{R}$ be any function with $w(a) = w(b) = 0$ and appropriate regularity conditions.

$I[y_0] \leq I[y_0 + tw]$, $\forall t, \forall w : [a,b] \to \mathbb{R}$ ← Perturbation idea

$$\left.\frac{d}{dt} I[y_0 + tw]\right|_{t=0} = 0 \quad \leftarrow \text{as } y_0 \text{ is minimizer}$$

DCT/Leibniz rule to swap diffn & $\int$: (continuity of L & its partial deriv. wrt t)

$$0 = \left.\frac{d}{dt}\int_a^b L(x, y_0 + tw, y_0' + tw')\, dx\right|_{t=0} = \int_a^b \underbrace{\frac{\partial L}{\partial y}(x, y_0+tw, y_0'+tw')w(x) + \frac{\partial L}{\partial z}(x, y_0+tw, y_0'+tw')w'(x)}_{= \frac{d}{dt} L(x, y_0+tw, y_0'+tw') \text{ [Chain rule]}} dx$$

$t = 0$ plugged in

$$\Rightarrow 0 = \int_a^b \frac{\partial L}{\partial y} w(x) + \underbrace{\left[\frac{\partial L}{\partial z} w(x)\right]_a^b}_{=0, \text{ as } w(a) = w(b) = 0} - \int_a^b \frac{d}{dx}\left[\frac{\partial L}{\partial z}\right] w(x)\, dx \quad \text{[integration by parts]}$$

$$\Rightarrow 0 = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial z}\right)\right] w(x)\, dx, \ \forall w$$

▨

$$\overset{\text{FLCV}}{\Rightarrow} \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial z}\right) = 0$$

- Fermat's Principle of Optics: Light follows the trajectory that takes the shortest time to travel.

$\Rightarrow$ Can use variational calculus & E-L eqns to derive laws of reflection & refraction.
Snell's Law

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- Example 1 Solution: (Shortest Path)

$$L(x,y,z) = \sqrt{1+z^2}\,, \quad \min_{\substack{y(x):\\ y(a)=y_1,\, y(b)=y_2}} I[y] = \int_a^b L(x,y,y')\,dx$$

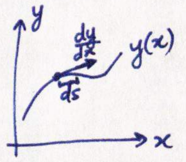
$$\frac{\partial L}{\partial y} = 0\,, \quad \frac{\partial L}{\partial z} = \frac{z}{\sqrt{1+z^2}}$$

Euler-Lagrange equations: $\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial z}\right) = 0 \Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$

$$\Rightarrow \frac{y''}{(1+(y')^2)^{3/2}} = 0 \quad \Rightarrow \underline{\underline{y''=0\,,\ y(a)=y_1,\ y(b)=y_2}}$$

So, $y(x)$ is a straight line connecting (a, y_1) and (b, y_2).

- Remark: $\frac{y''}{(1+(y')^2)^{3/2}}$ is the signed curvature $\begin{cases} \text{- 0 curvature corresponds to lines.} \\ \text{- constant curvature is a circle.}\end{cases}$



Clairaut's Thm [cont. second deriv]

$$\underbrace{\frac{d}{ds}\left(\frac{dy}{dx}\right)}_{\text{rate of change of gradient with } ds} \overset{\downarrow}{=} \frac{d}{dx}\left(\frac{dy}{ds}\right) = \frac{d}{dx}\left(\frac{dy}{\sqrt{dy^2+dx^2}}\right) = \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = \underbrace{\frac{y''}{(1+(y')^2)^{3/2}}}_{\text{curvature}}$$

(informal: L not dep on one of its variables $\Rightarrow$ something is constant/some deriv. is 0)

- Noether's Thm: Any differentiable symmetry of the action integral (or Lagrangian) has a corresponding conservation law.

① $L = L(x, z)$ independent of y.

$$\underbrace{\frac{\partial L}{\partial y} = 0}_{\text{differential symmetry}} \Rightarrow \text{By Euler-Lagrange eqns, } \frac{d}{dx}\left(\frac{\partial L}{\partial z}\right) = 0 \Rightarrow \underbrace{\boxed{\frac{\partial L}{\partial z} = \text{constant}}}_{\text{conservation law}}$$

eg: (Shortest Path)

$$L(x,y,z) = \sqrt{1+z^2} \Rightarrow \frac{\partial L}{\partial z} = \frac{y'}{\sqrt{1+(y')^2}} = \text{constant} \Rightarrow y' \text{ constant} \Rightarrow y \text{ linear.}$$

eg: (Conservation of Linear Momentum)

$$L(\underset{\text{time}}{t}, \underset{\text{position}}{x}, \underset{\text{velocity}}{x'}) = \underbrace{\tfrac{1}{2}m(x')^2}_{\text{kinetic energy}}$$

free particle (no potential field)
- minimize total kinetic energy $\int_{t_1}^{t_2} \frac{1}{2}m(x')^2\,dt$ given boundary cond.s $x(t_1)=x_1$ & $x(t_2)=x_2$

$$\frac{\partial L}{\partial z} = \boxed{m x'(t) = \text{constant}}$$

linear momentum conservation due to symmetry of physical laws in position.

② Thm: $L = L(y, z)$ independent of x $\Rightarrow$ BELTRAMI IDENTITY: $\boxed{y'\frac{\partial L}{\partial z}(y,y') - L(y,y') = \text{constant}}$

Pf: $$\frac{d}{dx}\left(y'\frac{\partial L}{\partial z} - L\right) = \underbrace{\frac{\partial L}{\partial y}y' + \frac{\partial L}{\partial z}y''}_{\text{chain rule}} - y''\frac{\partial L}{\partial z} - y'\frac{d}{dx}\left(\frac{\partial L}{\partial z}\right) = y'\underbrace{\left(\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial z}\right)\right)}_{=0 \text{ by Euler-Lagrange equation}} = 0 \quad \blacksquare$$

[continued.]

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② cont. eg: $L(y, z) = \frac{1}{2}mz^2 - V(y)$, $\frac{\partial L}{\partial z} = mz$

$\Rightarrow \underbrace{L(t, x, x')}_{\text{time, pos., velocity}} = \underbrace{\frac{1}{2}m(x')^2}_{\text{kinetic energy}} - \underbrace{V(x)}_{\text{potential energy}}$ ← Symmetry of physical laws with time

Beltrami: $x' \frac{\partial L}{\partial z}(x, x') - L(x, x') = m(x')^2 - \left(\frac{1}{2}m(x')^2 - V(x)\right) = \text{constant}$

$\Rightarrow \boxed{\underbrace{\frac{1}{2}m(x')^2 + V(x)}_{\text{Hamiltonian = total energy}} = \text{constant}}$ ← energy conservation

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- Hamilton's Principle: (The last example motivates Lagrangian mechanics.)

* A system in motion follows a trajectory that minimizes: $\underbrace{\int_{t_1}^{t_2} L(t, x, x')\, dt}_{\text{action integral}}$,

where the Lagrangian $L = T - V$. (T: kinetic energy, V: potential energy) (Hamiltonian is T+V)

- also called principle of least action (as we minimize action integral)
- solve E-L eqns: $\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) = 0$.

- Example 2 Solution: (Brachistochrone Problem)

$L(x, y, z) = \sqrt{\frac{1+z^2}{y}}$, $\min_{\substack{y(x): \\ y(0)=0,\, y(l)=h}} I[y] = \int_0^l \sqrt{\frac{1+(y')^2}{y}}\, dx$ ← can be derived from Hamilton's Principle after some massaging

(remove constant (>0))

L indep. of $x \Rightarrow$ Beltrami: $y' \frac{\partial L}{\partial z}(y, y') - L(y, y') = C$

$\Rightarrow \frac{(y')^2}{\sqrt{1+(y')^2}\sqrt{y}} - \frac{\sqrt{1+(y')^2}}{\sqrt{y}} = C$ ← constant

$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{k-y}{y}}$, $k = \frac{1}{C^2}$ (constant) ← hard to solve this ODE

Let $\tan(\phi) = \sqrt{\frac{y}{k-y}} \Rightarrow \frac{d\phi}{dx} = \frac{1}{2k\sin^2(\phi)}$ (ϕ func. of x) as $y = k\sin^2(\phi)$ & $\frac{d\phi}{dx} = \frac{d\phi}{dy}\frac{dy}{dx}$.

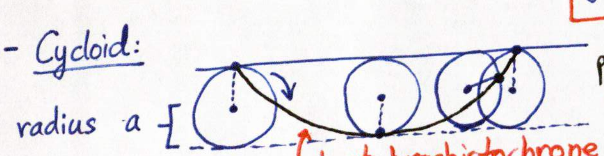
$\Rightarrow dx = d\phi\, 2k\sin^2(\phi)$ & $dy = d\phi\, 2k\sin(\phi)\cos(\phi)$

$\Rightarrow x = 2k\int \sin^2(\phi)\, d\phi = 2k\left(\frac{\phi}{2} - \frac{\sin(2\phi)}{4}\right) + c_1$ & $y = \int k\sin(2\phi)\, d\phi = \frac{-k\cos(2\phi)}{2} + c_2$ (constant of integration)

Boundary condition → $y(0) = 0 \Rightarrow \phi = 0$ & $x = 0 \Rightarrow c_1 = 0$ & $c_2 = \frac{k}{2}$

Let $k = 2a$ & $2\phi = \theta$. Then, $\boxed{\begin{aligned} x &= a(\theta - \sin(\theta)) \\ y &= a(1 - \cos(\theta)) \end{aligned}}$. ← Parametric equations of cycloid

- Cycloid:



- solution to brachistochrone problem
- solution to tautochrone problem (same period of oscillation of ball regardless of starting amplitude) → Christiaan Huygens

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- **Constrained Optimization:**

- What if we have constraints? $\max\limits_{y \in C} \int_a^b L(x,y,y')\,dx$ subj. to: $\underbrace{\int_a^b K(x,y,y')\,dx = K_0}_{\text{constraint}}$

  $L, K : [a,b] \times \mathbb{R} \times \mathbb{R}$
  (L: Lagrangian)
  $C = \{y : [a,b] \to \mathbb{R} \mid y(a) = y_1,\ y(b) = y_2,\ y \text{ twice differentiable}\}$

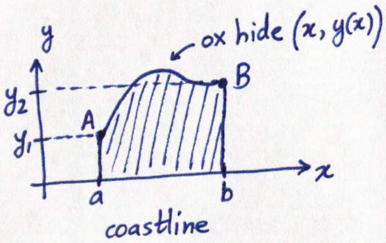
- Augmented Lagrangian: $L(x,y,y') + \lambda K(x,y,y')$ (λ: Lagrange multiplier)

  $$I[y] \triangleq \int_a^b L(x,y,y') + \lambda K(x,y,y')\,dx$$

  $\max\limits_{y \in C} I[y] \Rightarrow$ Euler-Lagrange Equations: $\left(\frac{\partial L}{\partial y} + \lambda \frac{\partial K}{\partial y}\right) - \frac{d}{dx}\left(\frac{\partial L}{\partial z} + \lambda \frac{\partial K}{\partial z}\right) = 0$,

  & explicitly impose $\int_a^b K(x,y,y')\,dx = K_0$

- Example 3: (Dido's Isoperimetric Problem)
  Legend is that around 850 B.C., Dido (Queen of Carthage) purchased land from a local king in the North African coastline that could be enclosed by the hide of an ox. The area she enclosed became the city of Carthage.



Want: Max area given arc length of A to B fixed.

$$\left.\begin{aligned} L(x,y,z) &= y \\ K(x,y,z) &= \sqrt{1+z^2} \end{aligned}\right\} \quad \max_{y \in C} \int_a^b L(x,y,y')\,dx = \underbrace{\int_a^b y(x)\,dx}_{\text{area}}$$

$$\text{s.t.} \quad \int_a^b K(x,y,y')\,dx = \int_a^b \sqrt{1+(y')^2}\,dx = K_0 \quad \text{(fixed length)}$$

(K_0: length of hide)

equivalent to minimizing arc length for fixed area

Augmented Lagrangian $= L(x,y,y') + \lambda K(x,y,y')$

Euler-Lagrange eqns: $\left(\frac{\partial L}{\partial y} + \lambda \frac{\partial K}{\partial y}\right) - \frac{d}{dx}\left(\frac{\partial L}{\partial z} + \lambda \frac{\partial K}{\partial z}\right) = 0$

$$\Rightarrow 1 - \frac{d}{dx}\left(\lambda \frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

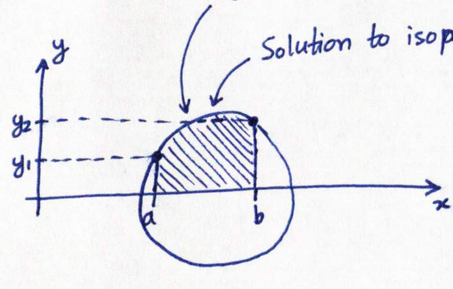
$$\Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = \frac{1}{\lambda}$$

$$\Rightarrow \frac{y''}{(1+(y')^2)^{3/2}} = \text{constant}$$

Constant curvature $\Rightarrow$ Solution is circle:

Choose circle so that this length is K_0 & area is maximized

Solution to isoperimetric problem



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